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The treatment of forces in Bloch analysis

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A R T I C L E I N F O

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A B S T R A C T

In periodic lattice structures, wave propagation on the infinite domain can be greatly simplified by invoking the Floquet–Bloch theorem. The theorem allows a system's degrees of freedom to be reduced to a small subset contained in a repeating unit cell. The equations of motion governing this subset contain internal force terms, which must be eliminated before establishing the eigenvalue problem for the dispersion relationships. There are subtle issues with regard to the elimination of these forces, which we address in this paper. We demonstrate that for any two- or three-dimensional periodic lattice, the internal forces vanish when acted upon by the linear transformation engendered by the degree of freedom reduction.

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1. Introduction

Lattice structures are widely used in applications where engineers require low weight and high stiffness [1]. In addition, periodic structures can have advantageous acoustic wave propagation characteristics. Mead provides an early use [2] and review [3] of Bloch analysis for investigating harmonic wave propagation in periodic systems. More recently, Phani et al. [4] derived the acoustic band structure of example honeycomb structures using finite element techniques and Bloch analysis. In these works and others, researchers use the fact that the equations of motion can be reduced to the minimum number of degrees of freedom (e.g., displacements) using Bloch analysis. While this is true for the displacements, it is not necessarily true for the forces accompanying the displacements. In his seminal work, Langley [5] properly poses the Bloch treatment of the unit cell forces for the case of a rectangular lattice. He shows that the Bloch procedure results in a zero vector for the final, reduced forces. Others (for e.g., Ref. [4]), cite Langley for use in their general geometry analysis, although Langley's treatment is valid for the rectangular case only. Still others incorrectly invoke the Bloch conditions on corner forces, however, without detriment to their analysis and results.

Here, we demonstrate that for any two- or three-dimensional lattice, the Bloch procedure results in a zero vector for the reduced forces acting on the unit cell. Consequently, the forces are no longer of concern.

2. Bloch analysis

Any lattice structure in a three-dimensional space can be constructed by translating a repeating unit cell along three independent basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ [6]. If $\mathbf{R}(\mathbf{r}_j)$ denotes the displacement of a point located at $\mathbf{r}_j$ in a chosen reference unit cell, then

$$\mathbf{R}(\mathbf{r}_j) = \mathbf{R}_j e^{(i\omega t - \mathbf{k}\cdot\mathbf{r}_j)}, \tag{1}$$

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in which $\mathbf{R}_j$ is the amplitude, ω is the frequency and $\mathbf{k}$ is the wave vector of the plane wave. The integer triplet (n_1, n_2, n_3) identifies the cell obtained by n_1, n_2 and n_3 translates of the unit cell along the $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ direction, respectively, such that the point in the cell (n_1, n_2, n_3) corresponding to $\mathbf{r}_j$ is located at $\mathbf{r}_j^{n_1,n_2,n_3} = \mathbf{r}_j + n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$. Letting $\mu_x = -\mathbf{k}\cdot\mathbf{e}_1$, $\mu_y = -\mathbf{k}\cdot\mathbf{e}_2$ and $\mu_z = -\mathbf{k}\cdot\mathbf{e}_3$, the displacement at $\mathbf{r}_j^{n_1,n_2,n_3}$ takes the form

$$\mathbf{R}(\mathbf{r}_j^{n_1,n_2,n_3}) = \mathbf{R}(\mathbf{r}_j)\mathrm{e}^{-\mathbf{k}.(n_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3)} = \mathbf{R}(\mathbf{r}_j)\mathrm{e}^{n_1\mu_x+n_2\mu_y+n_3\mu_z}. \tag{2}$$

Thus for two cells adjacent along the $\mathbf{e}_1$-axis, $\mathbf{R}(\mathbf{r}_j^{n_1,n_2,n_3}) = \mathbf{R}(\mathbf{r}_j^{n_1-1,n_2,n_3})\mathrm{e}^{\mu_x}$. We use this property and the like of that along the $\mathbf{e}_2$- and $\mathbf{e}_3$-axis to reduce the number of coordinates. In what follows, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ will be identified with x-, y- and z-axes, which are not assumed to be orthogonal.

3. Equations of motion

After invoking Lagrangian or Newtonian dynamics, the equations of motion for a general unit cell assume the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}, \tag{3}$$

in which matrices $\mathbf{M}$ and $\mathbf{K}$ represents the global mass and stiffness matrix of the unit cell, $\mathbf{q}$ and $\ddot{\mathbf{q}}$ represents the (nodal) displacements and accelerations, and $\mathbf{F}$ denotes the (nodal) forces. For plane harmonic waves, $\ddot{\mathbf{q}}$ can be replaced by $-\omega^2\mathbf{q}$ so that the equations of motion can be rewritten as

$$(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{q} = \mathbf{F}. \tag{4}$$

For plane waves in a periodic lattice, the Bloch analysis reduces the number of displacements in Eq. (4), i.e., we can write

$$\mathbf{q} = \mathbf{T}\hat{\mathbf{q}}. \tag{5}$$

In a planar lattice structure, $\mathbf{T}$ is a linear transformation parametrized by μ_x and μ_y. For example, in the case of a square honeycomb (Fig. 1), if $\mathbf{q}_{sq}$ denotes the displacements of the unit cell, we have $\mathbf{q}_{sq} = \mathbf{T}_{sq}\hat{\mathbf{q}}_{sq}$ in which

$$\mathbf{q}_{sq} = \begin{bmatrix} \mathbf{q}_i \\ \mathbf{q}_B \\ \mathbf{q}_T \\ \mathbf{q}_L \\ \mathbf{q}_R \\ \mathbf{q}_{LB} \\ \mathbf{q}_{RB} \\ \mathbf{q}_{LT} \\ \mathbf{q}_{RT} \end{bmatrix}, \quad \mathbf{T}_{sq} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}\mathrm{e}^{\mu_y} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}\mathrm{e}^{\mu_x} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}\mathrm{e}^{\mu_x} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}\mathrm{e}^{\mu_y} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}\mathrm{e}^{\mu_x+\mu_y} \end{bmatrix}, \quad \hat{\mathbf{q}}_{sq} = \begin{bmatrix} \mathbf{q}_i \\ \mathbf{q}_B \\ \mathbf{q}_L \\ \mathbf{q}_{LB} \end{bmatrix}. \tag{6}$$

Note that $\mathbf{q}_{RB} = \mathrm{e}^{\mu_x}\mathbf{q}_{LB}$ holds because $\mathbf{q}_{RB}$ is the common point for two adjacent cells: the cells located at $n_1\mathbf{e}_1 + n_2\mathbf{e}_2$ and $(n_1+1)\mathbf{e}_1 + n_2\mathbf{e}_2$ (see Fig. 2), i.e., $\mathbf{q}_{RB}^{n_1,n_2} = \mathbf{q}_{LB}^{n_1+1,n_2}$. By the Floquet–Bloch theorem, $\mathbf{q}_{LB}^{n_1+1,n_2} = \mathrm{e}^{\mu_x}\mathbf{q}_{LB}^{n_1+1,n_2}$. These two equations and similar relations yield (6). In a general case (not square honeycomb), Eq. (5) would result in an equation of motion in the form of $(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{T}\hat{\mathbf{q}} = \mathbf{F}$. If the external force term on the right-hand side of this equation could be eliminated, it would establish an eigenvalue problem yielding ω. We show that a matrix $\overline{\mathbf{T}}$ obtained from $\mathbf{T}$ by replacing μ_x, μ_y with $-\mu_x$, $-\mu_y$ will have the property that $\overline{\mathbf{T}}^{\mathrm{T}}\mathbf{F} = \mathbf{0}$, where $\overline{\mathbf{T}}^{\mathrm{T}}$ denotes the transpose of $\overline{\mathbf{T}}$. For example, for the special case

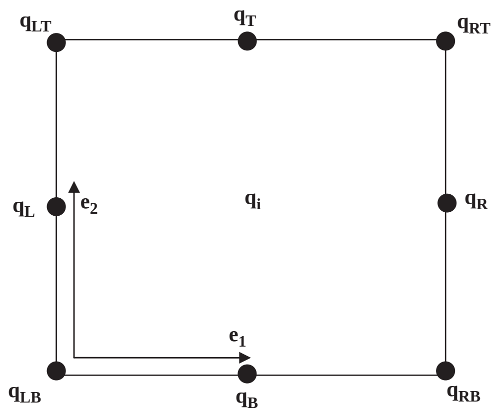

Fig. 1. Depiction of the nine displacements in a square honeycomb lattice.

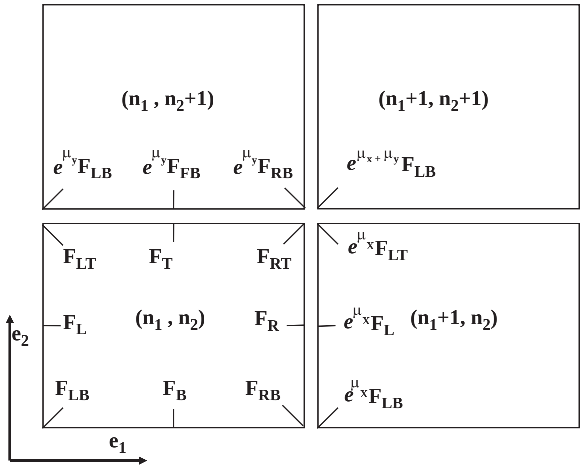

Fig. 2. Square honeycomb with $\mathbf{e}_1$, $\mathbf{e}_2$ as the coordinate unit vectors and the cells located at (n_1, n_2), (n_1+1, n_2), (n_1, n_2+1) and (n_1+1, n_2+1).

of the squarehoneycomb:

$$\overline{\mathbf{T}}_{sq}^{\mathrm{T}}\mathbf{F}_{sq} = \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_B + \mathrm{e}^{-\mu_y}\mathbf{F}_T \\ \mathbf{F}_L + \mathrm{e}^{-\mu_x}\mathbf{F}_R \\ \mathbf{F}_{LB} + \mathrm{e}^{-\mu_x}\mathbf{F}_{RB} + \mathrm{e}^{-\mu_y}\mathbf{F}_{LT} + \mathrm{e}^{-\mu_y-\mu_y}\mathbf{F}_{RT} \end{bmatrix}. \tag{7}$$

Note that others apply the Bloch approach to the forces in Eq. (7), and combine it with the equilibrium condition for the remaining terms in $\overline{\mathbf{T}}_{sq}^{\mathrm{T}}\mathbf{F}_{sq}$, and arrive at the equations,

$$\mathbf{F}_T = -\mathrm{e}^{\mu_y}\mathbf{F}_B, \quad \mathbf{F}_R = -\mathrm{e}^{\mu_y}\mathbf{F}_L, \tag{8}$$

$$\mathbf{F}_{RB} = -\mathrm{e}^{\mu_x}\mathbf{F}_{LB}, \quad \mathbf{F}_{LT} = -\mathrm{e}^{\mu_y}\mathbf{F}_{LB}, \ \ \mathbf{F}_{RT} = \mathrm{e}^{\mu_y+\mu_y}\mathbf{F}_{LB}, \tag{9}$$

resulting in the desired outcome $\overline{\mathbf{T}}_{sq}^{\mathrm{T}}\mathbf{F}_{sq} = \mathbf{0}$. This procedure cannot be followed, however. Langley has shown that Eq. (9) violates the power flow assumption through the square lattice structure. Here, we show by example that Eq. (9) does not hold. However, we do show that $\overline{\mathbf{T}}^{\mathrm{T}}\mathbf{F} = \mathbf{0}$ holds in general for cases such as triangular honeycomb, hexagonal honeycomb and general three-dimensional structures.

4. Force reduction

In this section it will be shown that $\overline{\mathbf{T}}^{\mathrm{T}}\mathbf{F} = \mathbf{0}$ holds for planar lattices. The three-dimensional case borrows similar arguments expanded in the remaining direction. In any lattice structure a minimal set of displacements can be defined, which we term $\mathbf{q}_i$ and $\widetilde{\mathbf{q}}$ for the internal and the cell boundary displacements, respectively. Applying the Bloch theorem, the remaining nodal displacements in the unit cell can be determined by pushing $\widetilde{\mathbf{q}}$ forward in a combination of the x or/and y directions. We term these mappings $\mathbf{T}_x$, $\mathbf{T}_y$ and $\mathbf{T}_{xy}$. Consequently, the relation between the minimum set of displacements $\mathbf{q}_i$, $\widetilde{\mathbf{q}}$ and the totality of displacements $\mathbf{q}$ is stated as

$$\mathbf{q} = \mathbf{T}\begin{bmatrix} \mathbf{q}_i \\ \widetilde{\mathbf{q}} \end{bmatrix}, \tag{10}$$

in which

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{T}_x \\ \mathbf{0} & \mathbf{T}_y \\ \mathbf{0} & \mathbf{T}_{xy} \end{bmatrix}, \tag{11}$$

such that $\mathbf{q}$ takes the form

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_i \\ \tilde{\mathbf{q}} \\ \tilde{\mathbf{q}}_x \\ \tilde{\mathbf{q}}_y \\ \tilde{\mathbf{q}}_{xy} \end{bmatrix}. \tag{12}$$

For example, in the two-dimensional square honeycomb (Fig. 1), if $\mathbf{q}^{sq}$ denotes a rearrangement of the displacements $\mathbf{q}_{sq}$, we have

$$\mathbf{q}^{sq} = \begin{bmatrix} \mathbf{q}_i^{sq} \\ \tilde{\mathbf{q}}^{sq} \\ \tilde{\mathbf{q}}_x^{sq} \\ \tilde{\mathbf{q}}_y^{sq} \\ \tilde{\mathbf{q}}_{xy}^{sq} \end{bmatrix} = \mathbf{T}^{sq} \begin{bmatrix} \mathbf{q}_i^{sq} \\ \tilde{\mathbf{q}}^{sq} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{T}_x^{sq} \\ \mathbf{0} & \mathbf{T}_y^{sq} \\ \mathbf{0} & \mathbf{T}_{xy}^{sq} \end{bmatrix} \begin{bmatrix} \mathbf{q}_i^{sq} \\ \tilde{\mathbf{q}}^{sq} \end{bmatrix}, \tag{13}$$

in which

$$\mathbf{q}_i^{sq} = \mathbf{q}_i, \quad \tilde{\mathbf{q}}^{sq} = \begin{bmatrix} \mathbf{q}_B \\ \mathbf{q}_L \\ \mathbf{q}_{LB} \end{bmatrix},$$

$$\tilde{\mathbf{q}}_x^{sq} = \begin{bmatrix} \mathbf{q}_R \\ \mathbf{q}_{RB} \end{bmatrix}, \quad \tilde{\mathbf{q}}_y^{sq} = \begin{bmatrix} \mathbf{q}_T \\ \mathbf{q}_{LT} \end{bmatrix}, \quad \tilde{\mathbf{q}}_{xy}^{sq} = [\mathbf{q}_{RT}], \tag{14}$$

and

$$\mathbf{T}_x^{sq} = \begin{bmatrix} \mathbf{0} & \mathbf{I}e^{\mu_x} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}e^{\mu_x} \end{bmatrix}, \quad \mathbf{T}_y^{sq} = \begin{bmatrix} \mathbf{I}e^{\mu_y} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}e^{\mu_y} \end{bmatrix}, \quad \mathbf{T}_{xy}^{sq} = [\mathbf{0} \;\; \mathbf{0} \;\; \mathbf{I}e^{\mu_x+\mu_y}]. \tag{15}$$

Implementing Bloch analysis, we can derive all the displacements of a cell from the minimal set of displacements $[\mathbf{q}_i \; \tilde{\mathbf{q}}]^{\mathrm{T}}$. The equation of motion at a cell located at (n_1, n_2) in the general two-dimensional lattice structure takes the form

$$(-\omega^2\mathbf{M} + \mathbf{K}) \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{T}_x \\ \mathbf{0} & \mathbf{T}_y \\ \mathbf{0} & \mathbf{T}_{xy} \end{bmatrix} \begin{bmatrix} \mathbf{q}_i \\ \tilde{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_i \\ \mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2} \\ \mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1,n_2} \\ \mathbf{F}_{\tilde{\mathbf{q}}y}^{n_1,n_2} \\ \mathbf{F}_{\tilde{\mathbf{q}}xy}^{n_1,n_2} \end{bmatrix}, \tag{16}$$

where $\mathbf{F}_{\tilde{\mathbf{q}}y}^{n_1,n_2}$ denotes the force collocated with $\tilde{\mathbf{q}}_y$ and applied on the cell (n_1, n_2). Similarly, $\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1-1,n_2}$ denotes the forces on cell $(n_1 - 1, n_2)$ collocated with $\tilde{\mathbf{q}}$, where $\tilde{\mathbf{q}}$ always references cell (n_1, n_2) herein. Note that since $\tilde{\mathbf{q}}$ is a vector of boundary displacements shared by two or more adjacent cells, the force nomenclature must also specify the translates n_1, n_2.

The translation matrices $\mathbf{T}_x$, $\mathbf{T}_y$ and $\mathbf{T}_{xy}$ have a special property: only one element in each row is nonzero because each displacement $\mathbf{q}$ in $\tilde{\mathbf{q}}_x$,$\tilde{\mathbf{q}}_y$,$\tilde{\mathbf{q}}_{xy}$ is obtained by shifting one element in $\tilde{\mathbf{q}}$. Also, only one element in each column is nonzero since each q shows up once in $\tilde{\mathbf{q}}_x$, $\tilde{\mathbf{q}}_y$ and $\tilde{\mathbf{q}}_{xy}$. We define negative translation matrices $\mathbf{T}_{-x}$, $\mathbf{T}_{-y}$ and $\mathbf{T}_{-xy}$ by replacing e^{μ_x}, e^{μ_y}, $e^{\mu_x+\mu_y}$ with $e^{-\mu_x}$, $e^{-\mu_y}$ and $e^{-\mu_x-\mu_y}$. Due to the aforementioned properties of $\mathbf{T}_x$, $\mathbf{T}_y$, $\mathbf{T}_{xy}$ it can be easily verified that

$$\mathbf{T}_\nu \mathbf{T}_{-\nu}^{\mathrm{T}} = \mathbf{I}, \quad \nu = x, y, xy, \tag{17}$$

where $\mathbf{I}$ denotes an identity matrix and $\mathbf{T}_{-\nu}^{\mathrm{T}}$ denotes the right inverse matrix of $\mathbf{T}_\nu$. As such, if $\mathbf{T}_\nu$ represents a pushing forward action in the ν direction, then $\mathbf{T}_{-\nu}^{\mathrm{T}}$ is a pulling back action in the same ν direction. Now, we consider $\mathbf{T}_{-\nu}^{\mathrm{T}}\mathbf{T}_\nu$ which is a diagonal matrix with diagonal elements of zero or one. By virtue of the Bloch procedure, $\mathbf{T}_x$,$\mathbf{T}_y$ and $\mathbf{T}_{xy}$ map $\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2}$ into $\mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1+1,n_2}$, $\mathbf{F}_{\tilde{\mathbf{q}}y}^{n_1,n_2+1}$ and $\mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1+1,n_2+1}$, respectively. The dimension of $\mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1+1,n_2}$ is less than or equal to the dimension of $\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2}$. For example in the case of the square honeycomb lattice depicted in Fig. 2

$$\mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1+1,n_2} = \mathbf{T}_x^{sq}\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2} = \begin{bmatrix} \mathbf{0} & \mathbf{I}e^{\mu_x} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}e^{\mu_x} \end{bmatrix} \begin{bmatrix} \mathbf{F}_B \\ \mathbf{F}_L \\ \mathbf{F}_{LB} \end{bmatrix} = \begin{bmatrix} e^{\mu_x}\mathbf{F}_L \\ e^{\mu_x}\mathbf{F}_{LB} \end{bmatrix}. \tag{18}$$

Note that $\mathbf{T}_{-x}^{\mathrm{T}}\mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1+1,n_2}$ pulls back the elements of $\mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1+1,n_2}$ in the x direction. $\mathbf{T}_{-x}^{\mathrm{T}}\mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1+1,n_2}$ has the same dimension as $\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2}$, but it is not the same vector. $\mathbf{T}_{-x}^{\mathrm{T}}$ recovers those elements of $\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2}$ which were pushed forward by $\mathbf{T}_x$. In other words, $[\mathbf{T}_{-x}^{\mathrm{T}}\mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1+1,n_2}]$ is the force on the cell (n_1, n_2) collocated with $\tilde{\mathbf{q}}_x^{n_1-1,n_2}$. For example in Fig. 2:

$$(\mathbf{T}_{-x}^{sq})^{\mathrm{T}}\mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1+1,n_2} = (\mathbf{T}_{-x}^{sq})^{\mathrm{T}}\mathbf{T}_{x}^{sq}\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I}\mathrm{e}^{-\mu_x} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}\mathrm{e}^{-\mu_x} \end{bmatrix}\begin{bmatrix} \mathrm{e}^{\mu_x}\mathbf{F}_L \\ \mathrm{e}^{\mu_x}\mathbf{F}_{LB} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_L \\ \mathbf{F}_{LB} \end{bmatrix},$$

with the same argument, $\mathbf{T}_{-xy}^{\mathrm{T}}\mathbf{T}_{xy}\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2}$ is the force on the cell (n_1, n_2) collocated with $\tilde{\mathbf{q}}_{xy}^{n_1-1,n_2-1}$. In the case of Fig. 3

$$(\mathbf{T}_{-x}^{sq})^{\mathrm{T}}\mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1,n_2} = (\mathbf{T}_{-x}^{sq})^{\mathrm{T}}\mathbf{T}_{x}^{sq}\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1-1,n_2} = (\mathbf{T}_{-x}^{sq})^{\mathrm{T}}\mathbf{T}_{x}^{sq}\begin{bmatrix} \mathbf{0} \\ \mathrm{e}^{-\mu_x}\mathbf{F}_R \\ \mathrm{e}^{-\mu_x}\mathbf{F}_{RB} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathrm{e}^{-\mu_x}\mathbf{F}_R \\ \mathrm{e}^{-\mu_x}\mathbf{F}_{RB} \end{bmatrix}.$$

We ultimately wish to show that

$$\overline{\mathbf{T}}^{\mathrm{T}}\mathbf{F} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{T}_{-x}^{\mathrm{T}} & \mathbf{T}_{-y}^{\mathrm{T}} & \mathbf{T}_{-xy}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{F}_i \\ \mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2} \\ \mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1,n_2} \\ \mathbf{F}_{\tilde{\mathbf{q}}y}^{n_1,n_2} \\ \mathbf{F}_{\tilde{\mathbf{q}}xy}^{n_1,n_2} \end{bmatrix} = \mathbf{0}. \tag{19}$$

Note that $\overline{\mathbf{T}}^{\mathrm{T}}\mathbf{F}$ consists of two vectors, $\mathbf{F}_i$ which is zero in the absence of external forces, and

$$\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2} + \mathbf{T}_{-x}^{\mathrm{T}}\mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1,n_2} + \mathbf{T}_{-y}^{\mathrm{T}}\mathbf{F}_{\tilde{\mathbf{q}}y}^{n_1,n_2} + \mathbf{T}_{-xy}^{\mathrm{T}}\mathbf{F}_{\tilde{\mathbf{q}}xy}^{n_1,n_2}. \tag{20}$$

Since $\mathbf{F}_{\tilde{\mathbf{q}}x}^{n_1,n_2} = \mathbf{T}_x^{\mathrm{T}}\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1-1,n_2}$ (and similar identities), we can restate Eq. (20) as

$$\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2} + \mathbf{T}_{-x}^{\mathrm{T}}\mathbf{T}_x\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1-1,n_2} + \mathbf{T}_{-y}^{\mathrm{T}}\mathbf{T}_y\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2-1} + \mathbf{T}_{-xy}^{\mathrm{T}}\mathbf{T}_{xy}\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1-1,n_2-1}. \tag{21}$$

As was shown in Fig. 3, $\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2}$, $\mathbf{T}_{-x}^{\mathrm{T}}\mathbf{T}_x\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1-1,n_2}$, $\mathbf{T}_{-y}^{\mathrm{T}}\mathbf{T}_y\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1,n_2-1}$ and $\mathbf{T}_{-xy}^{\mathrm{T}}\mathbf{T}_{xy}\mathbf{F}_{\tilde{\mathbf{q}}}^{n_1-1,n_2-1}$ are the forces on the cells (n_1, n_2), (n_1-1, n_2), (n_1, n_2-1) and (n_1-1, n_2-1) collocated with $\tilde{\mathbf{q}}$, $\tilde{\mathbf{q}}_x^{n_1-1,n_2}$, $\tilde{\mathbf{q}}_y^{n_1,n_2-1}$ and $\tilde{\mathbf{q}}_{xy}^{n_1-1,n_2-1}$, respectively. As it can be seen for the example given in Fig. 4, the equilibrium condition in the region between adjacent cells collocated with $\tilde{\mathbf{q}}^{n_1,n_2}$(shaded region of Fig. 3) results in Eq. (20) evaluating to zero. Consequently, the equation of motion (4) would take

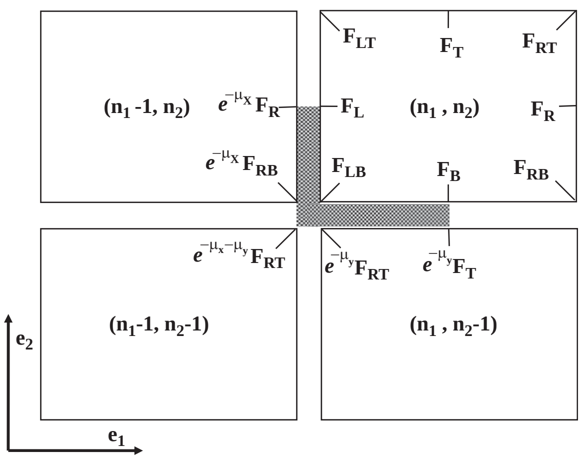

Fig. 3. Square honeycomb with the cells located at (n_1, n_2), (n_1-1, n_2), (n_1, n_2-1) and (n_1-1, n_2-1). The equilibrium condition for the shaded area results in Eq. (19).

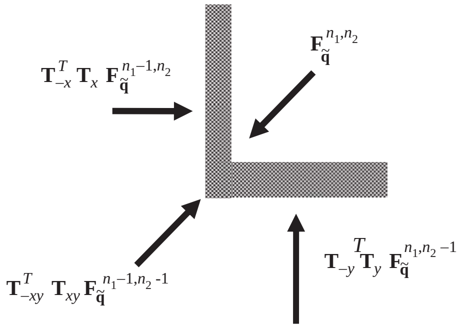

Fig. 4. Depiction of the equilibrium condition in the region between adjacent cells collocated with $\tilde{\mathbf{q}}^{n_1,n_2}$.

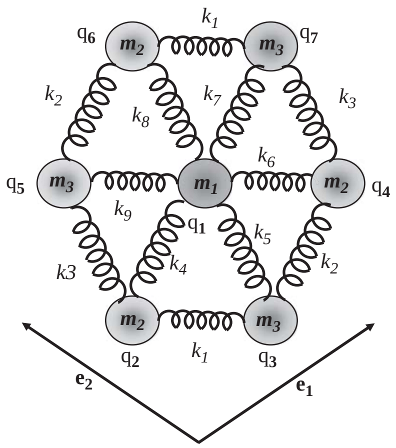

Fig. 5. A hexagonal honeycomb lattice with an internal degree of freedom and forces.

Table 1
Parameters chosen for the hexagonal honeycomb.

m_1	m_2	m_3	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9
1	2	3	10	20	30	40	50	10	20	30	10

the form

$$\bar{\mathbf{T}}^{\mathrm{T}}(-\omega^2\mathbf{M}+\mathbf{K})\mathbf{T}\begin{bmatrix}\mathbf{q}_i\\ \tilde{\mathbf{q}}\end{bmatrix}=\bar{\mathbf{T}}^{\mathrm{T}}\mathbf{F}=\mathbf{0}, \tag{22}$$

which is the desired eigenvalue problem parameterized by ω.

5. Analysis of free wave motion in an example

In this example, we consider a hexagonal honeycomb lattice structure (Fig. 5). Each mass has one degree of freedom; it can vibrate in a direction orthogonal to the plane spanned by $\mathbf{e}_1$, $\mathbf{e}_2$ and the springs exert forces outward or inward to this plane. The internal degree of the freedom and the internal forces are modelled by the internal mass m_3 and springs k_4 to k_9. The arbitrary values for the k's and m's are tabulated in Table 1. Eqs. (10)–(12) are applied here in the same manner. For this case:

$$\mathbf{q}_i=[\mathbf{q}_1],\quad \tilde{\mathbf{q}}=\begin{bmatrix}\mathbf{q}_2\\ \mathbf{q}_3\end{bmatrix},\quad \tilde{\mathbf{q}}_x=[\mathbf{q}_4],\quad \tilde{\mathbf{q}}_y=[\mathbf{q}_5],\quad \tilde{\mathbf{q}}_{xy}=\begin{bmatrix}\mathbf{q}_6\\ \mathbf{q}_7\end{bmatrix},$$

$$\mathbf{T}_x=[\mathbf{I}e^{\mu_x}\;\;\mathbf{0}],\quad \mathbf{T}_y=[\mathbf{0}\;\;\mathbf{I}e^{\mu_y}],\quad \mathbf{T}_{xy}=\begin{bmatrix}\mathbf{I}e^{\mu_x+\mu_y} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}e^{\mu_x+\mu_y}\end{bmatrix}.$$

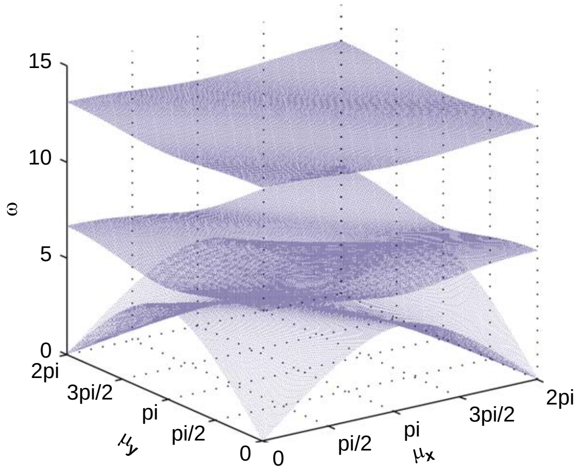

Fig. 6. Three ω's versus μ_x and μ_y in the case of the mass–spring hexagonal lattice.

Table 2
Comparison between the force $\mathbf{F}_6$, the external force on the cell collocated with $\mathbf{q}_6$, and the force obtained by direct use of Bloch analysis $e^{\mu_x+\mu_y}\mathbf{F}_2$.

$\mathbf{F}_6$	−21	1	−9	17	11	3	10	7	16	14
$e^{\mu_x+\mu_y}\mathbf{F}_2$	−31	−13	−33	6	−7	−16	−8	−8	−5	2
Difference	10	14	24	11	18	19	18	15	21	12

As it was proved for the general case, Eq. (19) holds in this case. Utilizing Eq. (22), the dispersion curves for the values of Table 1 for the hexagonal honeycomb were found (Fig. 6).

Also external forces on the unit cell collocated with $\mathbf{q}_6$ was obtained for 10 random values of $\tilde{\mathbf{q}}$ and $\mathbf{q}_i$ and it is compared with $e^{\mu_x+\mu_y}\mathbf{F}_2$ for $\mu_x = \mu_y = 0.5$. These numbers are shown in Table 2. It is apparent from this table that the Bloch analysis is not valid for the forces in this case.

6. Concluding remarks

Using a preferred ordering of the Bloch transformation matrix, we demonstrate that for any two- or three-dimensional lattice, the Bloch procedure results in a zero vector for the reduced forces. This establishes the validity of the final eigenvalue problem authors have posed to calculate dispersion relationships in general, periodic lattices.

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